

The Effect of Prepayment Penalties on Subprime Borrowers' Decisions to Default: A Perfect Storm

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Abstract

We present a continuous time option pricing model for mortgage valuation to examine the effects of a prepayment penalty on the default and the prepayment decision of subprime borrowers. We show that the options embedded in the mortgage contract have significant positive values for the mortgage borrower. In particular, the value of the prepayment option for the subprime mortgage borrower is significant. The major finding from this study is that the prepayment penalty affects not only the prepayment decision but also the default decision of subprime borrowers, by reducing the value of these options. The prepayment penalty increases the likelihood of default by subprime borrowers and makes refinancing prohibitively costly. These two effects induced by the prepayment penalty generate a high rate of mortgage defaults and foreclosures. Consequently, borrower defaults began to increase as house prices started to depreciate in 2006.

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1 Introduction

We are in the middle of an economy-wide financial crisis with the subprime mortgage market at the epicenter of the crisis. The first sign of trouble was a sudden increase of default in the subprime mortgage market with a drastic house price depreciation in 2006. What caused this sudden increase in default rates in the subprime mortgage market after many years of high prepayment rates? This paper is an attempt to answer this question. This is of interest to academics, policy makers, and the general public.

We present a continuous time version of an option pricing model for mortgage valuation to examine the effects of a prepayment penalty on the default and the prepayment decisions of subprime borrowers. Our study shows that a sudden increase of default in the subprime market is caused by the prepayment penalty, that is prevalent in subprime mortgage contracts, along with house price depreciation. These two together have generated a perfect storm in subprime mortgage market.

This paper demonstrates that the prepayment penalty has two effects on the subprime borrower's mortgage termination decisions. The prepayment penalty makes prepayment or refinancing prohibitively costly. Equity extraction from house price appreciation is the key to the subprime mortgage design. And this equity extraction from house price appreciation is only possible through the prepayment. With a high prepayment penalty, this prepayment option is not available to the subprime mortgage borrowers.

The options embedded in mortgage contracts add economic value to the mortgages since these options provide partial protection from the volatility of house prices. In particular, the value of the prepayment option for the subprime mortgage borrower is significant. The

prepayment penalty increases the likelihood of default by subprime borrowers because the penalty reduces the option values of the mortgages. By these two effects of the prepayment penalty, the subprime borrowers become more vulnerable to default caused by the house price depreciation. This inability to refinance mortgage loans and the high default point, induced by the prepayment penalty translated into a high rate of mortgage defaults and foreclosures. Consequently, borrower default began to increase as house price started to depreciate in 2006.

The paper is organized as follows. In section 2, we consider mortgages as financial assets that generate dividends in the form of the housing services. We characterize a mortgage as a fixed income derivative with two options- default and prepayment. We formulate a general mortgage valuation model as a stochastic control problem in continuous time and reformulate the problem as an optimal stopping problem reducing it to a free-boundary problem. We then derive the Hamiltonian-Jacobi-Bellman equation for the stochastic control problem. In section 3, we consider the simplest case where the mortgage contract only has the default option. We derive an explicit functional form for optimal default and other important expressions related to mortgages such as loan to value (LTV) ratio, yield and recovery rate (RR). We also show how optimal default is affected by the parameters. In section 4, we consider the case when the borrowers also have a prepayment option, as this is essential to the analysis of the subprime mortgages. In section 5, we investigate the option values embedded in the mortgage contract to show how these option values affect the default decision of mortgage borrowers. It turns out that the default and prepayment decisions are both closely related to the value of options. Because one of the distinctive characteristics of subprime mortgages is the prevalence of prepayment penalties, we include the prepayment penalty in section 6. We analyze how its inclusion affects the default decision of subprime borrowers. Section 7 concludes.

2 Mortgage valuation

The most prominent contractual feature of mortgage loans, compared to other loan contracts, are the length of maturity and the possibility of early termination, through either prepayment or default. A mortgage lender does not just lend money but also gives the borrower the right and the option to default or to prepay. Thus the borrower can decide whether to continue the mortgage contract by paying the periodic payment or to terminate the contract by exercising the given options. This possibility of early termination of the mortgage contract becomes a serious risk to the lender. Quantifying and managing these risks is essential for the mortgage lender. In this section, we present a mortgage valuation model that captures these unique features of the mortgage contract.

The financial asset, the house generates housing services, x , as dividends. Housing services can be interpreted as the difference between rental income and other expenses owning house. We assume that the housing service follows a geometric Brownian motion with drift.

$$dx = \alpha x dt + \sigma x dz, \quad (1)$$

where α is the growth rate of housing services, σ is the volatility parameter of the housing services, and dz is the increment of the standard Wiener process. We capture the uncertain future stochastic economic environment or the underlying source of uncertainty by this stochastic process.

The price of housing equals the expected present value of future housing services.

$$P(x(t)) = \mathbb{E}_t \left[\int_{\tau=t}^{\infty} e^{-\rho(\tau-t)} x(\tau) d\tau \right] = \frac{x(t)}{\rho - \alpha}, \quad (2)$$

where \mathbb{E}_t denotes the expectation based on the information as of time t , and ρ denotes an exogenous discount rate. We set $x(0)$, the value of housing services at the mortgage origination date, equal to 1.

We consider the mortgage as a derivative asset whose value is derived from the value of the underlying asset, the house. The price of a house is determined by the housing services from equation (2). Thus the value of mortgage $M(x(t))$ is also determined by the housing services.

Given the housing price in equation (2), the house owner minimizes her mortgage liability by choosing an optimal time to exercise termination options. This is equivalent to maximize her equity since her equity on housing is the difference between housing price and the mortgage liability.¹

$$E(x(t)) = P(x(t)) - M(x(t)) \quad (3)$$

We formulate the home owner's problem as a stochastic control problem in continuous time and reformulate the problem as an optimal stopping problem and reduce the problem to a free-boundary problem. In general, optimal stopping problems are two-dimensional in the sense that they consist of finding the unknown value function and the unknown optimal boundaries (or unknown expiration date T) simultaneously; the value function can be seen as a function of unknown stopping boundaries. We find the solution in reverse order: First, we find the free-boundaries of the differential equation. Second, we find the optimal stopping level of the state variable, that is, we determine the optimal default point and the optimal prepayment point. Finally, we solve the initial continuous time stochastic control problem.

We assume that the termination of contract occurs solely for financial considerations. We

¹Home equity is the current market value of a house minus the outstanding mortgage balance.

denote two thresholds as x_{**} and x^* : the default point x_{**} , is the level of housing services where the house owner optimally exercises the default option; prepayment point x^* , is the level of housing services where the house owner optimally exercises the prepayment option.² Thus these options are at the money at x_{**} and at x^* respectively.

We define the optimal stopping time of default as

$$T(x_{**}) = \inf \{ t \geq 0, x(t) \leq x_{**} \}$$

and the optimal stopping time of prepayment as

$$T(x^*) = \inf \{ t \geq 0, x(t) \geq x^* \}$$

The random time variables $T(x_{**})$ and $T(x^*)$ denote the first passage times that the housing services x hits the down-barrier x_{**} and up-barrier x^* , respectively. Let's define a random variable $T = T(x_{**}) \wedge T(x^*)$ as the first time the process $x(t)$ reaches default point x_{**} or prepayment point x^* . Thus T is the optimal stopping time of the mortgage contract, i.e., the optimal termination time of mortgage contract.

The home owner's decision problem is to choose the level of x where she optimally exercises the options and terminates the mortgage contract. She has to decide whether the future expected gain from maintaining the mortgage contract will outweigh the loss due to terminating mortgage contract. Therefore her optimization is with respect to the choice of threshold values x_{**} and x^* .³ The home owner is confronted with the following optimization problem.

² x_{**} is a down barrier and x^* is a up barrier for this stochastic process. The two barriers are so called absorbing barriers, since the process x is killed as soon as it hits one of the barriers. Therefore, the mortgage is a knock-out type double barrier option.

³ The advantage of this dynamic programming over the other methods is that it is possible to use this approach when the constraints are stochastic but usually the solution is a nonlinear PDE.

The principle of optimality implies that E satisfies the Bellman equation.

$$E(x) = \sup_{x_{**}, x^*} \left\{ \begin{aligned} & \mathbb{E}_x \left[\int_{\tau=0}^T e^{-\rho\tau} (x(\tau) - c) d\tau \right] \\ & + \mathbb{E}_x \left[e^{-\rho T} \mid x(T) = x_{**} \right] E(x_{**}) Pr[x(T) = x_{**}] \\ & + \mathbb{E}_x \left[e^{-\rho T} \mid x(T) = x^* \right] E(x^*) Pr[x(T) = x^*] \end{aligned} \right\} \quad (4)$$

$$\text{subject to } dx = \alpha x dt + \sigma x dz \quad \text{and } x(0) = 1,$$

where x is a state variable for this stochastic control problem and $E(x)$ denote the expected discounted equity from following an optimal policy given the initial state $x(0) = 1$. Thus, the equity function $E(x)$ is the value function. The first term in equation (4) is the expected present value of equity for the periods before the home owner terminates mortgage contract. The second term is the expected present value of equity when the home owner defaults. The last term is the expected present value of equity when the home owner prepays. $E(x)$ is the sum of three terms.

The default point x_{**} and prepayment point x^* are the optimal stopping levels of x . These thresholds or cutoff points x_{**} and x^* separate the whole range of x into three regions: region below x_{**} , region between x_{**} and x^* , and the region above x^* . The region between x_{**} and x^* is called inaction or continuation region where the continuation of payment is optimal and no option is exercised. Other two regions, the region below x_{**} and the region above x^* are optimal stopping or the mortgage termination region. In the region below x_{**} , exercising the default option is optimal and in the region above x^* exercising prepayment option is optimal for the house owner.

For the continuation region where the home owner doesn't exercise options, i.e., for the open interval (x_{**}, x^*) , or for the time periods $[0, T)$, we can make the probability of reaching either threshold arbitrarily small by setting dt sufficiently small.

Thus from equation (4), we have

$$E(x(t)) = \mathbb{E}_x \left[\int_{\tau=0}^t e^{-\rho\tau} (x(\tau) - c) d\tau \right] \quad \text{for } x \in (x_{**}, x^*). \quad (5)$$

Notice that there is no maximization operator on the right hand side since no action is taken, i.e., no option is exercised in the interval (x_{**}, x^*) . In this inaction region, the homeowner just pays a periodic payment c and receives the housing services x without exercising any mortgage option.

It follows from (5) and from the principle of optimality that

$$E(x) = (x - c) dt + e^{-\rho dt} \mathbb{E}_x [E(x')] \quad \forall x \in (x_{**}, x^*) \quad (6)$$

where x' is the next period housing services so $x' = x + dx$. We dropped time t from the equation since this equation is independent from time. The equation holds for all t as far as x is in (x_{**}, x^*) . Therefore the value function $E(x)$ is common to all $t \in [0, T)$. The current state $x(t)$ matters, but the calendar date t by itself has no effect.

The first term on the right-hand side is the immediate net housing service from holding the mortgage contract. The second term constitutes the continuation value of the mortgage holding. The important point here is that the home owner maximizes her equity considering not just the immediate payout, net housing service $(x - c) dt$, but also the future value of equity $\mathbb{E}_x [E(x')]$. Therefore, the homeowner's mortgage termination decision depends on the

net housing services and on the future expected value of equity.

By expanding the right hand side of (6) we have

$$E(x) = (x - c) dt + (1 - \rho dt + O(dt)) \mathbb{E}_x [E(x) + dE(x)],$$

where $O(dt)$ is the sum of higher-order terms in dt . We omit $O(dt)$ and rearrange terms in the equation to have following stochastic differential equation

$$\rho E(x) dt = (x - c) dt + \mathbb{E}[dE(x)]. \quad (7)$$

The left hand side of equation (7) measures the normal return that the home owner would require for holding the house. The right hand side of the equation measures the expected total return from holding the house, that is, the sum of net housing services and the expected rate of capital appreciation. This is also the opportunity cost of exercising an option today. Thus, this equality is the return equilibrium condition⁴. These two must be equal, otherwise the mortgage would be improperly valued. The equality becomes a no-arbitrary or equilibrium condition, expressing the homeowner's willingness to hold the house.⁵

If we rearrange equation (7), we have

$$\rho E(x) dt + c dt = x dt + \mathbb{E}[dE(x)]. \quad (8)$$

The left hand side of equation is the marginal cost of holding the mortgage contract and the right hand side of the equation is the marginal benefit of holding mortgage contract. Thus, the equation shows that the marginal cost and the marginal benefit of holding a mortgage

⁴It is also called asset equilibrium condition

⁵The equity function $E(x)$ satisfies the stochastic differential equation by the martingale property.

contract should be same at the optimum.

Again we rearrange equation (7), we have

$$E(x) = \frac{1}{\rho} \left[(x - c) + \frac{1}{dt} \mathbb{E}[dE(x)] \right].$$

The first term of the right hand side of equation is an immediate payout or net dividends from the house and the second term is its expected rate of capital gain. Therefore the home equity is essentially the amount of ownership that has been built up by the holder of mortgage through the periodic payments and the house price appreciation.

By using Ito's lemma, we can rewrite the the second term of right-hand side in equation (8) as

$$\mathbb{E}[dE(x)] = \alpha x E'(x) dt + \frac{1}{2} \sigma^2 x^2 E''(x) dt. \quad (9)$$

The expected value of the change of future equity or capital appreciation depends not just on the trend of housing services, α , but also the volatility of housing services, σ , that is the source of option value in mortgages.

Substituting this equation (9) into equation (8) leads to the ordinary differential equation⁶

$$\frac{1}{2} \sigma^2 x^2 E''(x) + \alpha x E'(x) - \rho E(x) = -x + c. \quad (10)$$

This is the Hamilton-Jacobi-Bellman equation for the above stochastic continuous time dynamic programming problem (4). It is the necessary and sufficient condition for the optimum.⁷ The optimal value function E satisfies the equation (10) in the continuation region,

⁶This equation is a nonconstant (variable) coefficients, nonhomogeneous, second order linear differential equation. Also this type of differential equation is called a Cauchy-Euler equation.

⁷ This is the stochastic analog to a continuous-time Bellman equation. The difference is that the variance

the interval (x_{**}, x^*) .

This second order ordinary differential equation has the solution.

$$E(x) = e_1 x^{m_1} + e_2 x^{m_2} + \frac{x}{\rho - \alpha} - \frac{c}{\rho}, \quad (11)$$

$$\text{where } m_1, m_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) \mp \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\rho}}{\sigma^2}$$

are the two roots of the characteristic equation of the differential equation (10).

To complete the solution of this equation we have to specify two free boundaries, $E(x_{**})$ and $E(x^*)$ of the above equation. These two free boundaries, x_{**} and x^* are determined by requiring that E and E' be continuous at x_{**} and x^* . These conditions, called value matching and smooth pasting, reproduce the solution obtained by maximizing equation (4). In a free-boundary problem, the solution of the differential equation and the domain over which the differential equation must be solved, need to be determined simultaneously.

From the equations (2), (3) and (11), we can derive the explicit mortgage value function as the difference between house price and equity :

$$M(x) = P(x) - E(x) = -e_1 x^{m_1} - e_2 x^{m_2} + \frac{c}{\rho}, \quad (12)$$

Thus the house price , the equity, and the economic value of the mortgage are all functions of the housing service x .

term makes the HJB equation a second order ordinary difference equation, while the Bellman equation is first order.

3 Model with only default option

For this section we assume that the home owner does not have the prepayment option. When there is no prepayment option, the homeowner can terminate the mortgage contract only by defaulting. Thus the homeowner has a binary decision problem at every instant. She maximizes her equity over the binary choice: terminate the mortgage contract by exercising the default option or maintain the mortgage contract by continuing to pay the periodic payment c .

The equity function $E(x)$ satisfies the Bellman equation

$$E(x) = \max_{a \in \mathbb{A}} \left\{ E(x_*) , (x - c) dt + e^{-\rho dt} \mathbb{E} [E(x') | x] \right\} , \quad (13)$$

$$\text{subject to } dx = \alpha x dt + \sigma x dz \quad \text{and } x(0) = 1 ,$$

where $\mathbb{A} = \{ \text{exercise default option, pay periodic payment} \}$ is the set of home owner's possible actions and x_* denotes default point when we have only default option. We distinguish x_* with x_{**} , the default point when we have both options. Accordingly, we denote the termination payoff as $E(x_*)$ when there is only default option. The first term in the bracket is the equity when the home owner exercises the default option or the termination payoff. The second term is the equity when she does not exercise the option or the payoff when she postpones exercising the option.

Since $E(x)$ is increasing in x , the optimal policy of the home owner is

$$a^* = \begin{cases} \text{exercise default option} & \text{if } x \leq x_* \\ \text{pay periodic payment} & \text{if } x > x_* . \end{cases} \quad (14)$$

The threshold x_* separates the whole range of x into two regions: the region below x_* and the region above x_* . Therefore $x_* \leq x(0) = 1 \leq x^*$. For x lower than x_* , stopping payment is optimal and for x higher than x_* , continuing payment is optimal. That is, the region below x_* is the optimal stopping region and the region above x_* is the continuation region.

The solution of equation (13) can be written as

$$E(x) = E(x_*) \cdot \mathbb{1}_{[0, x_*]}(x) + \left\{ (x - c) dt + e^{-\rho dt} \mathbb{E} [E(x') | x] \right\} \cdot \mathbb{1}_{(x_*, \infty)}(x), \quad (15)$$

where $\mathbb{1}_A(x)$ is an indicator function. The first term, $E(x_*) \cdot \mathbb{1}_{[0, x_*]}(x) = 0$ in the continuation region, (x_*, ∞) . Thus

$$E(x) = (x - c) dt + e^{-\rho dt} \mathbb{E} [E(x') | x] \quad \text{for } (x_*, \infty). \quad (16)$$

As shown in the previous section, the solution of this equation is

$$E(x) = e_1 x^{m_1} + e_2 x^{m_2} + \frac{x}{\rho - \alpha} - \frac{c}{\rho}.$$

We have three unknowns in this case: two constants of integration, e_1 and e_2 from the equation, and a free boundary, i.e., the unknown default point x_* . To determine these three unknowns, we need three conditions that $E(x)$ must satisfy.

We have the first condition from the fact the equity can not grow to infinity. Thus the coefficient e_2 in the differential equation should be zero. Otherwise, $E(x)$ is not defined at the large value of x .

Thus when the default option is not exercised, i.e., for $x \in (x_*, \infty)$, or for $t \in [0, T)$, the

solution of the stochastic Bellman equation (13) is

$$E(x) = e x^{m_1} + \frac{x}{\rho - \alpha} - \frac{c}{\rho} \quad (17)$$

$$m_1 = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\rho}}{\sigma^2},$$

where m_1 is the negative root associated with the differential equation (10).

To determine the remaining two unknowns, the constant of integration e and a free boundary x_* , we need two more boundary conditions.

The first boundary condition is the value matching condition:

$$E(x_*) = e x_*^{m_1} + \frac{x_*}{\rho - \alpha} - \frac{c}{\rho} = 0. \quad (18)$$

This condition implies that the home owner defaults when the housing price equals the mortgage value, $P(x) = M(x)$. That is, she defaults when the value of equity on the mortgage becomes zero, $E(x) = 0$. This condition matches the value of the unknown function $E(x) = 0$ to those of the known termination payoff function $E(x_*)$. But the boundary x_* itself is an unknown.

The second boundary condition is the smooth pasting condition:

$$E'(x_*) = e m_1 x_*^{m_1 - 1} + \frac{1}{\rho - \alpha} = 0. \quad (19)$$

This boundary condition implies that the value of $E(x)$ and $E(x_*)$ should meet tangentially at the boundary x_* . The value matching and smooth-pasting conditions determine the free

boundary x_* that separates the continuation region from the stopping region.

With these boundary conditions (18) and (19), we can get the analytical expression of the constant of integration e and the default point x_* . That is, we determine the unknown functional form of $E(x)$ and the unknown default point x_* from equations (18) and (19) simultaneously.

$$e = \frac{1}{m_1(\alpha - \rho)} \left[\frac{c}{\rho} \left(\frac{m_1(\rho - \alpha)}{m_1 - 1} \right) \right]^{1 - m_1} \quad (20)$$

$$x_* = \left(\frac{c}{\rho} \right) \left[\frac{m_1(\rho - \alpha)}{m_1 - 1} \right]. \quad (21)$$

Using equations (20) and (21) we draw the house price $P(x)$, the mortgage value $M(x)$, the equity function $E(x)$, and the default point x_* for specific parameter values in figure (1).

We also derive the analytical expression of the loan to value ratio (LTV), recovery ratio (RR), and yield as functions of the parameters, α , σ , ρ , and c . All these variables x_* , LTV , RR , and $yield$ are endogenously determined.

$$\begin{aligned} LTV &= \frac{M(1)}{P(1)} = \left(l + \frac{c}{\rho} \right) (\rho - \alpha), \\ RR &= \frac{P(x_*)}{M(1)} = \left[\left(\frac{c}{\rho} \right) \left(\frac{m_1}{m_1 - 1} \right) \right] \left(l + \frac{c}{\rho} \right)^{-1}, \\ Yield &= \frac{c}{M(1)} = c \cdot \left(l + \frac{c}{\rho} \right)^{-1}, \end{aligned} \quad (22)$$

$$\text{where } l = \frac{1}{\rho - \alpha} x_*^{1 - m_1} - \left(\frac{c}{\rho} \right) x_*^{-m_1}.$$

The following table summarizes the comparative statics analysis results and shows how the endogenous variables, x_* , LTV , $Yield$, and RR are affected by changes in the exogenous parameters c , σ , α , and ρ ⁸.

⁸These results are all under the conditions, $\alpha > 0$, $\rho > 0$ and $\rho > \alpha$

Table 1: Comparative statics analysis of x_* , LTV , $Yield$, and RR

	Δx_*	ΔLTV	$\Delta Yield$	ΔRR
Δc	(+)	(+)	(+)	(+)
$\Delta \sigma$	(-)	(-)	(+)	(-)
$\Delta \alpha$	(-)	(-)	(-)	(?)
$\Delta \rho$	(+)	(+)	(+)	(+)

The signs (+) and (-) in the table show the signs of the partial derivatives of endogenous variables with respect to the parameters. For example, the (+) sign in first row and first column in the table shows the positive sign of partial derivative, $\frac{\partial x_*}{\partial c}$ and it can be interpreted as higher periodic payment c induces higher default point x_* . Similarly, the (-) sign in the table denotes the negative sign of the partial derivatives.

If we substitute equation (9) to (8), we have

$$\rho E(x) dt + c dt = x dt + \underbrace{\alpha x E'(x) dt + \frac{1}{2} \sigma^2 x^2 E''(x) dt}_{\mathbb{E}[dE(x)]} . \quad (23)$$

We can interpret the results of the comparative statics analysis with the equation above. If c increases, the left hand side of equation (23) becomes larger than the right hand side of the equation, x^* should be higher to equalize the marginal cost and the marginal benefit of holding mortgages implying $\frac{\partial x_*}{\partial c} > 0$. Similarly, if the discount factor ρ increases, the left hand side becomes greater than the right hand side, so the house owner defaults at higher x_* or defaults more quickly implying $\frac{\partial x_*}{\partial \rho} > 0$. If α increases, the expected future equity value that is a part of the economic value of equity $E(x)$ increases, so the house owner holds the mortgage longer, i.e., $\frac{\partial x_*}{\partial \alpha} < 0$. Thus if the house owner predicts future house price appreciation, she defaults at a lower x_* , that is, she holds the mortgage longer. Because future expected equity value is higher when σ is large, $\frac{\partial x_*}{\partial \sigma} < 0$. If the house owner expects volatile

future house price movement, she holds mortgage longer.⁹ Basically the exogenous parameters c , σ , α , and ρ affect not just immediate payoff $(x - c) dt$ but also the expected future value of equity $\mathbb{E}[dE(x)]$.

Initial mortgage rates in the subprime market are significantly higher than prime mortgage rates. While this is true for interest rates on fixed-rate mortgages (FRMs), it is also true, contrary to popular belief, on teaser rates of hybrid adjustable rate mortgages (ARMs)(Bhardwaj and Sengupta 2008). For numerical calculation we use $c = 1.75$ as periodic payment of subprime mortgages and $c = 1.25$ as periodic payment of prime mortgages. Since $\frac{\partial x_*}{\partial c} > 0$, subprime mortgage borrower defaults faster than prime mortgage borrower.

From the equations (21) and (22), we can calculate numerical values of x_* , LTV , $Yield$, and RR for various values of c and σ . For these calibrations, we assume that $\alpha = 0.03$, $\rho = 0.07$. Given values for parameters, α , σ , ρ , and c , we can get the numerical values of the default point x_* , LTV , $Yield$, and RR explicitly. We report this calibration results in table (2). We are able to confirm the results from our comparative statics analysis with this numerical exercise.

⁹In the continuation region, (x_*, ∞) , $E(x)$, $E'(x)$, and $E''(x)$ are all positive.

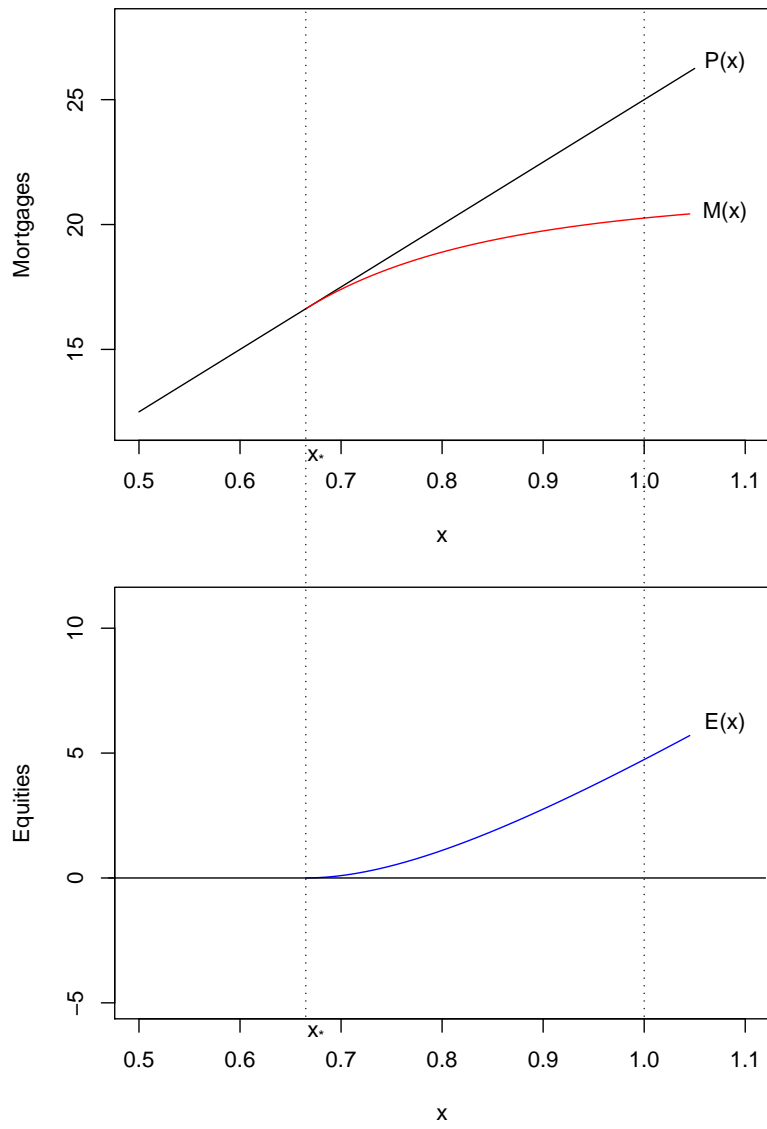
Table 2: x_* , LTV , $Yield$, and RR for various value of c and σ

$\sigma = 0.10$	c	x_*	$LTV(\%)$	$Yield(\%)$	$RR(\%)$
	1.00	0.500	57.09	7.01	87.59
	1.25	0.625	71.10	7.03	87.91
	1.50	0.750	84.28	7.12	88.99
	1.75	0.875	95.09	7.36	92.02
	2.00	1.000	100.00	8.00	100.00

$\sigma = 0.15$	c	x_*	$LTV(\%)$	$Yield(\%)$	$RR(\%)$
	1.00	0.443	56.38	7.10	78.65
	1.25	0.554	69.37	7.21	79.91
	1.50	0.665	81.04	7.40	82.07
	1.75	0.776	90.69	7.72	85.56
	2.00	0.887	97.40	8.21	91.05

$\sigma = 0.20$	c	x_*	$LTV(\%)$	$Yield(\%)$	$RR(\%)$
	1.00	0.389	54.72	7.31	71.15
	1.25	0.487	66.55	7.51	73.13
	1.50	0.584	77.06	7.79	75.78
	1.75	0.681	85.97	8.14	79.25
	2.00	0.779	92.95	8.61	83.76

Figure 1: $P(x)$, $M(x)$, $E(x)$, and x_* with only default option



4 Model with both options

Mortgage contracts are based on the borrower's ability to repay the mortgage loan. In the prime mortgage case this ability is based on well documented, proven future income. Thus, the prime mortgage lending is based on borrowers future income. However, the subprime mortgage design is based on the fact that the dominant form of wealth of low-income household is potentially their home equity since the subprime borrowers have a lower income than the prime borrowers and their ability to repay the mortgage loan is not proven, sometimes not documented properly. Thus, the subprime mortgages are closely linked to appreciation of the underlying asset, the house. No other consumer loan has the contractual feature that the borrowers ability to repay is so sensitively linked to appreciation of the underlying asset. The equity extraction from the house price appreciation is the key to the subprime mortgage contract design.¹⁰ And this equity extraction from house price appreciation is only possible through the borrower's prepayment option. Thus, the prepayment option is the integral part of the subprime mortgage design.

If borrowers prepay the current mortgage contract and move to a lower interest mortgage contract, they can pay a smaller periodic payment. This is the prepayment incentive to both prime and subprime mortgage borrowers. Subprime mortgage borrowers have another incentive to prepay mortgage: by their credit improvement, they can step up to prime borrower status. Even if there is no change in interest rate as we assumed here, subprime borrower has an incentive to prepay. Therefore including prepayment option in our model is crucial to examine the behavior of subprime borrowers.

In the previous section we assume that borrowers do not have the prepayment option.

¹⁰While equity extraction is common in the prime mortgage market, it is even more prevalent in the subprime mortgage market. Chomsisengphet and Pennington-Cross (2006) show that a higher proportion of subprime refinancing involve equity extraction, compared to prime refinancing.

Thus they have only two choices; being current on payment or default and terminate the mortgage contract. Therefore if they do not default on the contract, their payment is a perpetuity. This is a very unrealistic assumption, especially for subprime mortgages. In this section, the borrowers have three choices on their mortgage contract: pay the periodic payment, default or prepay. We show how the default point x_{**} is affected by the inclusion of the prepayment option in the model.

When the options are not exercised, i.e., for the continuation region $x \in (x_{**}, x^*)$, the solution of the Bellman equation (4) is

$$E(x) = e_1 x^{m_1} + e_2 x^{m_2} + \frac{x}{\rho - \alpha} - \frac{c}{\rho},$$

$$\text{where } m_1, m_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) \mp \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\rho}}{\sigma^2}.$$

When we have a prepayment option in the model, both constants of integration e_1 and e_2 are not zero. Thus we have four unknowns, the constants of integration of the equation e_1 , e_2 , the default point x_{**} and the prepayment point x^* . To determine these four unknowns, we need four boundary conditions: the value matching conditions and smooth pasting conditions at x_{**} and at x^* respectively.

The value matching condition at the default point x_{**} is

$$E(x_{**}) = 0 \tag{24}$$

and the smooth pasting condition at the default point x_{**} is

$$E'(x_{**}) = 0 . \quad (25)$$

These are the same conditions for the default point x_* in equations (18) and (19).

The value matching condition at the prepayment point x^* is

$$M(1) + E(x^*) + k_p = P(x^*) . \quad (26)$$

The left hand side of equation is the total cost of exercising prepayment option at x^* that is the sum of $M(1)$, the mortgage value at the origination date and $E(x^*)$, the equity at the prepayment point and k_p , the required prepayment penalty. The right hand side of equation (26) is the benefit from exercising prepayment option at x^* , the housing price $P(x^*)$. Thus the home owner exercises the prepayment option only when the benefit from exercising it exceeds the total cost of prepayment, that is, when the option is in the money.¹¹

The smooth pasting condition at the prepayment point x^* is

$$E'(x^*) = \frac{1}{\rho - \alpha} . \quad (27)$$

The smooth pasting condition is a necessary condition for optimal exercise of the options since the default point and prepayment point are free boundaries. It requires $M(x)$ to be tangent to $x/(\rho - \alpha)$ at x_{**} . And also that the slope of $E(x)$ at x^* should be equal to the slope of $P(x)$. Figure (2) shows that the house price $P(x)$, the mortgage value $M(x)$, the equity function $E(x)$, default point x_{**} , prepayment point x^* and these boundary conditions graphically.

¹¹Borrowers will prepay only when the value of their mortgages exceed their origination value $M(1)$ by at least k_p , i.e., $M(x^*) = M(1) + k_p$

From above four boundary conditions, we have following system of equations;

$$\begin{aligned}
e_1 x_{**}^{m_1} + e_2 x_{**}^{m_2} + \frac{x_{**}}{\rho - \alpha} - \frac{c}{\rho} &= 0 \\
m_1 e_1 x_{**}^{m_1 - 1} + m_2 e_2 x_{**}^{m_2 - 1} + \frac{1}{\rho - \alpha} &= 0 \\
e_1 x^*{}^{m_1} + e_2 x^*{}^{m_2} - e_1 - e_2 + k_p &= 0 \\
m_1 e_1 x^*{}^{m_1 - 1} + m_2 e_2 x^*{}^{m_2 - 1} &= 0 .
\end{aligned} \tag{28}$$

This is a system of four nonlinear equations with four unknowns, e_1 , e_2 , x_{**} , and x^* . The unknown functional form of $E(x)$ and the two unknown free boundaries are determined simultaneously by these system of equations. Since the system of equation (28) can not be solved analytically, we solve this system of equations numerically by the Newton-Raphson method.¹² We set $\rho = 0.07$ and $\alpha = 0.03$ for these numerical calculation. Before we consider the effect of prepayment penalty, we examine the effect of the inclusion of prepayment option in the model. Thus, at this point we set $k_p = 0$.

Now we can see the effects of inclusion of prepayment option to the model. The numerical calculations are reported in table (3).

The default point, x_{**} , in the model with the prepayment option, is lower than the default point, x_* , in the model without the prepayment option. Thus, the house owner holds the mortgage longer when we include the prepayment option in the model. The difference, $(x_* - x_{**})$, widens as the periodic payment c increases, since the prepayment option is more valuable to the subprime mortgages borrower. For example, when $c = 1.75$ and $\sigma = 0.2$, the percentage change in the default point is as high as 11.31%. The subprime borrowers have a bigger incentive for prepayment since they are paying a higher periodic payment c . After prepaying the mortgage, the subprime borrower can step up to a prime mortgage contract by

¹²The error tolerance for the approximation is 10^{-8} . The number of maximum iteration is 10^5 .

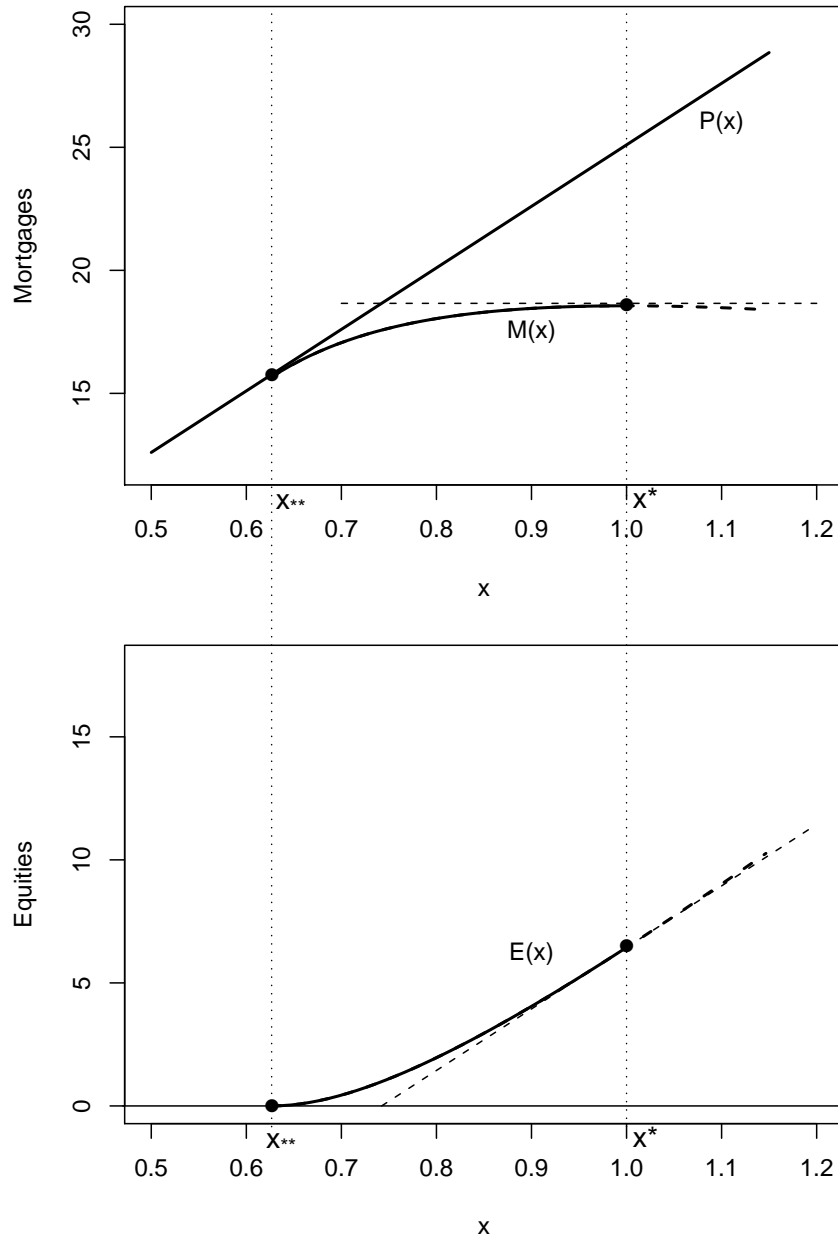
Table 3: The change in the default point for different c and σ

c	σ	x_*	x_{**}	% change
1.25	0.05	0.687	0.687	0.00
	0.10	0.625	0.620	0.80
	0.15	0.554	0.540	2.53
	0.20	0.487	0.466	4.31
1.50	0.05	0.824	0.823	0.12
	0.10	0.750	0.726	3.20
	0.15	0.665	0.627	5.71
	0.20	0.584	0.541	7.36
1.75	0.05	0.962	0.917	4.68
	0.10	0.875	0.803	8.23
	0.15	0.776	0.696	10.31
	0.20	0.681	0.604	11.31

the improvement of her credit rating. The prepayment option is more valuable to the subprime borrowers. The higher σ , the larger the difference, $(x_* - x_{**})$. When $c = 1.75$ and $\sigma = 0.05$, the percentage change in default point is 4.68% but it is 11.68% when $c = 1.75$ and $\sigma = 0.20$. In the model with only default option, the borrower never prepays since they do not have that option. However, in the model with a prepayment option, the borrower prepays whenever $x \geq 1$. In the next section, we investigate why and how the inclusion of the prepayment option in the mortgage contract affects the default decision of the mortgage borrowers.

Figure 2: $P(x)$, $M(x)$, $E(x)$, and x_{**} with both options

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5 Default decision and option values

In the previous sections, we show that the default point x_{**} is greatly affected by the inclusion of termination options. In this section we investigate how the options embedded in the mortgage contracts influence the borrower's mortgage default decision by calculating the economic values of the options embedded in the mortgage contract explicitly.

We define M^0 as the mortgage value without termination options, and $M(x)$, the mortgage value with termination options. If there is no termination option in the mortgage contract, the mortgage is a perpetuity. That is, the mortgage is just a fixed income security to the lender. Since the borrower pays a periodic payment c forever, M^0 is $\frac{c}{\rho}$. Notice that this mortgage value is independent to the level of housing services x and house price.

As we showed in the previous section, the mortgage value with termination options $M(x)$ is a function of the housing services x . The difference between the mortgage value without termination option M^0 and the mortgage value with termination option $M(x)$ is the option value $OV(x)$ embedded in the mortgage contract defined as

$$OV(x) = M^0 - M(x) \quad (29)$$

Since the option has a positive value to the borrower, the mortgage liability $M(x)$ decreases by the inclusion of the options. Rearranging the terms of (29) leads to

$$M(x) = M^0 - OV(x) . \quad (30)$$

When there is only a default option, the total option value, $OV(x)$, is just $DOV(x)$, the value

of the default option. We denote the mortgage value with the default option, $M^d(x)$, as

$$M^d(x) = M^0 - DOV(x) . \quad (31)$$

When there are both options, the total option value, $OV(x)$, is the sum of the default option value, $DOV(x)$, and the prepayment option value, $POV(x)$. We denote the mortgage value with both options, $M^p(x)$, as

$$M^p(x) = M^0 - (DOV(x) + POV(x)) . \quad (32)$$

The borrower does not want to have a mortgage liability bigger than the market value of house, which would imply negative home equity. That is, a home owner defaults her mortgage contract when her home equity becomes zero. Since the home equity is the difference between the house price and the mortgage liability, a home owner makes her default decision by comparing the price of her house and the value of the mortgage. Thus, given the house price, the home owner's default decision depends on the mortgage value.

As in equation (30), the mortgage value is the difference between the mortgage value without termination options M^0 and the option values embedded in the mortgage contract. Since the mortgage value without termination options, M^0 , is given by the specified periodic payment c , the mortgage value depends on the option value $OV(x)$. Thus given c , the periodic payment, the borrowers default decision is determined by the option values.

To get the numerical option values, we derive the three different equity functions from the above three different types of mortgage values.

The equity without termination options can be written as

$$E^0(x) = P(x) - M^0 = \frac{x}{\rho - \alpha} - \frac{c}{\rho}. \quad (33)$$

Figure 3: Mortgage value and equity without option

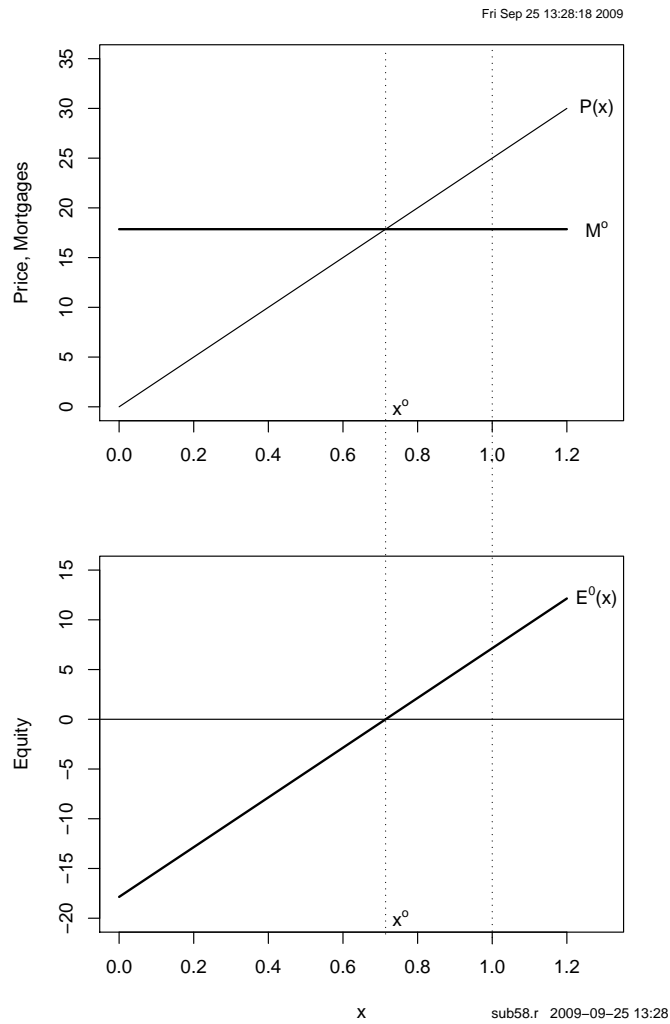


Figure (3) shows the mortgage value without options M^0 and the equity without options $E^0(x)$. We denote x^0 as the level of housing service where the equity without termination option is zero, i.e., $E^0(x^0) = 0$. At x^0 , the house price, $P(x)$, equals M^0 , the mortgage value

without termination options. Thus $x^0 = \left(\frac{\rho-\alpha}{\rho}\right) \cdot c$. Notice that $E^0(x)$ can be negative if $x < x^0$. The option value depends on σ , the volatility of housing service and the housing price as well as α and ρ . However x^0 depends on the parameters α and ρ but not on σ since it is unrelated to the option.

The equity with termination options is the sum of $E^0(x)$ and the option values embedded in mortgage contract. From equations (12) and (30), we have

$$E(x) = E^0(x) + OV(x) . \quad (34)$$

We denote $E^P(x)$ as the equity when borrowers have both the default option and the prepayment option:

$$E^P(x) = e_1 x^{m_1} + e_2 x^{m_2} + \frac{x}{\rho-\alpha} - \frac{c}{\rho} . \quad (35)$$

$E^d(x)$ is the equity when borrowers have only the default option.

$$E^d(x) = e x^{m_1} + \frac{x}{\rho-\alpha} - \frac{c}{\rho} . \quad (36)$$

The first two plots in figure (4) shows the different mortgage functions M^0 , M^d , and M^P , as well as the three different equity functions E^0 , E^d , and E^P . The distinction of the above three different equity functions E^0 , E^d , and E^P is critical to measure the values of the default and prepayment options. We can express the option values in terms of equity functions, $E^0(x)$, $E^d(x)$, and $E^P(x)$. We derive the explicit expressions for the option values from the equations (33) – (36).

The value of the default option is the difference between the two equity functions $E^d(x)$

and $E^0(x)$:

$$\begin{aligned} DOV(x) &= E^d(x) - E^0(x) \\ &= e x^{m_1} . \end{aligned} \tag{37}$$

The value of the prepayment option is the difference between the two equity functions $E^p(x)$ and $E^d(x)$:

$$\begin{aligned} POV(x) &= E^p(x) - E^d(x) \\ &= (e_1 - e) x^{m_1} + e_2 x^{m_2} . \end{aligned} \tag{38}$$

The value of both options is the sum of two values.

$$\begin{aligned} OV(x) &= POV(x) + DOV(x) \\ &= e_1 x^{m_1} + e_2 x^{m_2} . \end{aligned} \tag{39}$$

All these option values are functions of σ as well as α and ρ . Figure (4) shows the values of these options. The value of the default option is the vertical distance between the two equity functions $E^d(x)$ and $E^0(x)$ (between M^0 and $M^d(x)$) and the value of prepayment option is the vertical distance between the two equity functions $E^p(x)$ and $E^d(x)$ (between $M^d(x)$ and $M^p(x)$). Figure (4) shows the following:

1. As the housing services, x , increases, the default option value decreases exponentially, since the probability of using the default option decreases as x increases.
2. As the housing services, x , increases, the prepayment option value increases since the probability of using the prepayment option increases.
3. As the housing services, x , increases, the total option value decreases. Since the fall of the default option value offsets the increase of the prepayment option value.

Given parameter values, we can calculate numerical option values from equation (37)–(39). Table (4) shows the calculated values of both options and the mortgage values for different value of c and σ at x^0 . The numbers in the parentheses in the default column and the prepayment column show the option values as a percentage of the total option value. We list the mortgage values at x^0 in the last column of the table. In this table, we can see that mortgage values are affected by changes in the option value. In turn, these changes in mortgage value affect the default decision of the borrower. The values of both options increase as σ increases since the option is a partial protection against the fluctuation of x and p . Thus, the higher σ , the larger the option values, the smaller mortgage liability, and the lower x_{**} . Thus, $E(T(x_{**}))$, the expected optimal stopping time is larger when σ is higher. That is, the borrower holds the mortgage longer on average when σ is bigger. For example, when $c = 1.75$, if the volatility σ increases from 0.05 to 0.2, $OV(x)$, the option value increases from 1.404 to 6.503. The option value with $\sigma = 0.2$ is about four times larger than the option value with $\sigma = 0.05$. Accordingly, the mortgage liability of the borrower decreases from 23.596 to 18.497. Thus the borrower defaults at $x_{**} = 0.604$ instead at $x_{**} = 0.917$. That is, $E(T(x_{**}))$, the expected optimal stopping(default) time is larger. This shows a cascade effect, the default decision depends on the mortgage value, the mortgage value depends on the option values, and the option values depends on the volatility of the housing services. Thus, the default decision depends exclusively on the volatility of housing services and the housing price through the option values via the mortgage values.

This table also shows that the inclusion of the prepayment option in mortgage contract gives additional value to the mortgage contract. As a result the home owner will choose to exercise the default option at a lower x , i.e., $x_{**} < x_*$. For example, when $c = 1.75$ and $\sigma = 0.1$, the option value is 1.227 and the borrower defaults at $x_* = 0.875$ when there is no prepayment option. However, the prepayment option adds 1.984 to the total option value and

Table 4: The value of options and the mortgages for different c and σ at x_0

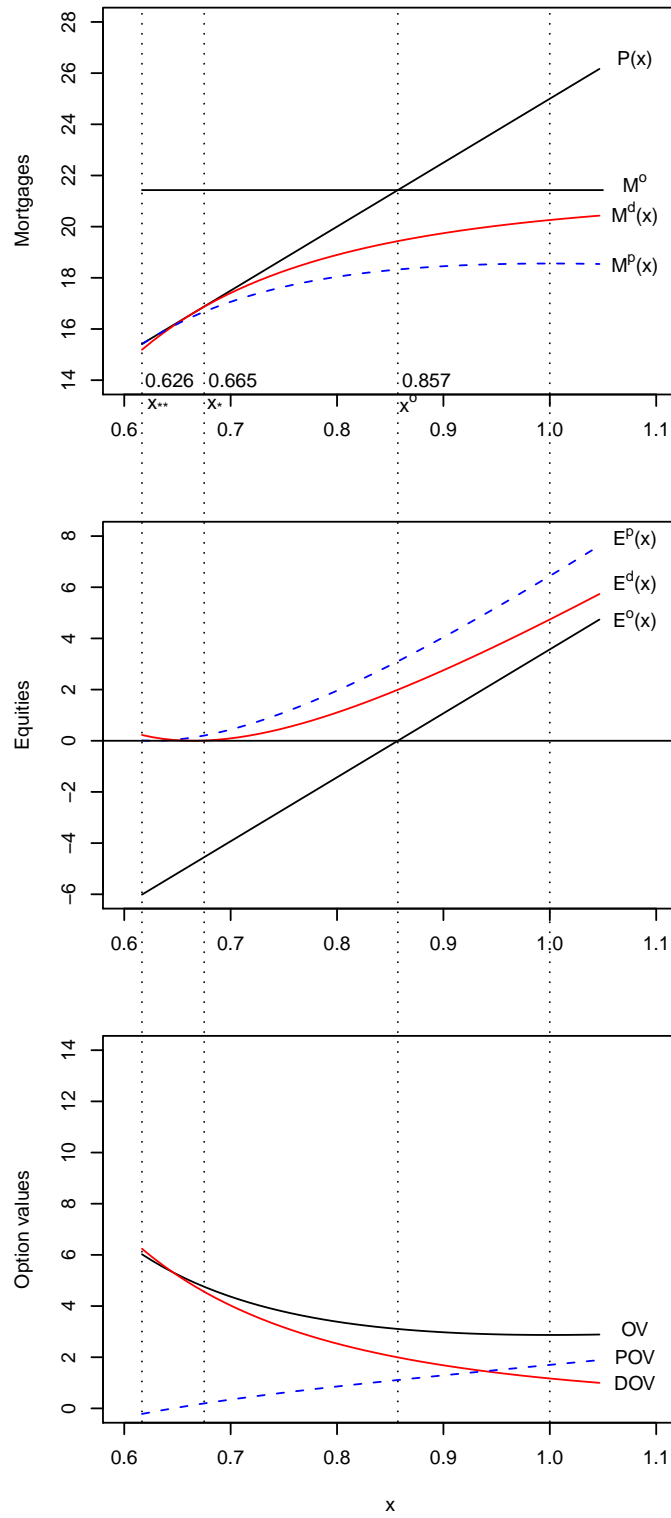
c	σ	x_0	x_*	x_{**}	option values			$MP(x^0)$
					default (%)	prepayment(%)	both	
1.25	0.05	0.714	0.687	0.687	0.255 (100.0)	≈ 0 (0.00)	0.255	17.602
	0.10		0.625	0.620	0.877 (89.9)	0.099 (10.1)	0.976	16.880
	0.15		0.554	0.540	1.662 (81.4)	0.381 (18.6)	2.043	15.814
	0.20		0.487	0.466	2.506 (79.0)	0.667 (21.0)	3.173	14.685
1.50	0.05	0.857	0.824	0.823	0.306 (90.6)	0.032 (9.4)	0.339	21.090
	0.10		0.750	0.726	1.052 (65.5)	0.555 (34.5)	1.607	19.822
	0.15		0.665	0.627	1.994 (64.2)	1.110 (35.8)	3.104	18.324
	0.20		0.584	0.541	3.007 (66.4)	1.523 (33.6)	4.530	16.899
1.75	0.05	1.000	0.962	0.917	0.358 (25.5)	1.046 (74.5)	1.404	23.596
	0.10		0.875	0.803	1.227 (38.2)	1.984 (61.8)	3.211	21.789
	0.15		0.776	0.696	2.327 (47.2)	2.603 (52.8)	4.930	20.070
	0.20		0.681	0.604	3.508 (53.9)	2.995 (46.1)	6.503	18.497

the borrower defaults at the lower $x_{**} = 0.803$. The existence of the prepayment option affects the value of mortgages even if interest rates are taken to be nonstochastic and constant, as is the case here.

As we have mentioned, the equity extraction is very important for subprime borrowers and that equity extraction is possible only through prepayment. Thus, this prepayment option value is prominent in the subprime mortgage case. Table 4 shows this fact clearly. When $c = 1.25$ and $\sigma = 0.1$ the prepayment option value is only 0.099 that is 10.1% of the total option value. But when $c = 1.75$ and $\sigma = 0.1$, the prepayment option value is 1.984, or 61.8% of the total option value. Thus in the subprime mortgage case, the prepayment option adds a significant value to the mortgage.

Figure 4: Value of options

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6 Model with prepayment penalty

Subprime borrowers are protected from large losses from decreases in property values because of the default option and the low level of their equity. With the default option, they have a limited liability instead of a unlimited liability for the loan. This means that when a bad event occurs, like a drastic housing price depreciation, they can exercise the default option for hedging purposes. But lenders, on the other hand, are subjected to big default losses if house values drop. When house prices rise, borrowers can prepay and extract the accumulated equity from the appreciation. They can use the prepayment option for speculation purposes as usual option exercising practice. As shown in the previous section, the prepayment option gives a substantial economic value to the subprime mortgage borrower. But due to the prepayment option, lenders are unlikely to benefit from any gains if the house prices rise. For the lender the prepayment option given to the borrower becomes an another source of risk, prepayment risk, associated with the early unscheduled return of principal. Therefore, lenders are likely to lose money whether house prices increase or decrease, as lenders are subject to two risks: default risk and prepayment risk.

Because of default risk, lenders require very large periodic payments for subprime mortgage borrowers. In order to mitigate the prepayment risk, subprime lenders typically include prepayment penalties or fees as part of the mortgage contract. One of the distinctive characteristics of subprime mortgages, relative to the prime mortgages, is the size and prevalence of the prepayment penalties.¹³ These penalties are seldom imposed in the conventional mortgage market. From an economic standpoint, prepayment penalties can be thought as a premium added to mortgages to compensate the lender for the supposedly high prepayment risk generated by the subprime loan. The lender uses prepayment penalties to prevent prepayment

¹³See, e.g., Farris and Richardson (2004). Fannie Mae estimates that 80 percent of subprime mortgages have prepayment penalties, while only two percent of prime mortgages have prepayment penalties (see Zigas, Parry, and Weech (2002)).

and to offset losses from prepayment. Thus subprime mortgages typically have significantly higher periodic payments and higher prepayment penalties than prime mortgages, in which case they are typically zero.

If there is no prepayment penalty or fee, a house owner could make a profit out of the mortgage contract by prepaying the mortgage whenever x is higher than 1. In this section, we include a positive prepayment penalty, $k_p > 0$, to avoid this unrealistic prediction of the model. Thus, the subprime mortgage lenders impose positive penalties on prepayments.

However, there is a maximum prepayment penalty k_p^* that the lender can impose to the borrowers. The borrower prepays only when the total cost of prepayment, $M(1) + k_p$, is less than the benefit from prepayment, $M(x^*)$.¹⁴ Thus, $M(1) + k_p \leq M(x^*)$. Since the maximum of $M(x^*)$ is $\frac{c}{\rho}$, the upper limit for k_p is

$$k_p^* = \frac{c}{\rho} - M(1) . \quad (40)$$

If $k_p^* > \frac{c}{\rho} - M(1)$, the borrower would never prepay ($x^* = \infty$). The table (5) shows the upper limit values of k_p for different values of c and σ .

Table 5: k_p^* , the upper bound of the prepayment penalty for different values of c and σ

c	$\sigma = 0.05$	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$
1.25	0.000	0.083	0.518	1.221
1.50	0.006	0.358	1.169	2.163
1.75	0.358	1.227	2.327	3.508

To examine the effect of the prepayment penalty on the borrowers mortgage termination decisions, we solve the system of equations (29) numerically and report the optimal default

¹⁴ $M(1) = M(x(0))$ is the origination value of the mortgage, i.e., the initial mortgage value at $t = 0$. It is the book value of the mortgage.

points x_{**} , the optimal prepayment points x^* , and the option values for the different periodic payment c and the prepayment penalty k_p in table (6). The bold numbers in the second column of the table shows the the upper bound of k_p for different values of c . Figure (5) shows that the mortgage functions $M(x)$, with and without the prepayment penalty, when $c = 1.75$ and $\sigma = 0.15$. As shown in the table and the figure, the prepayment penalty affects both on prepayment decision and also on default decision.

Table 6: Default points, prepayment points, and option values with the prepayment penalties

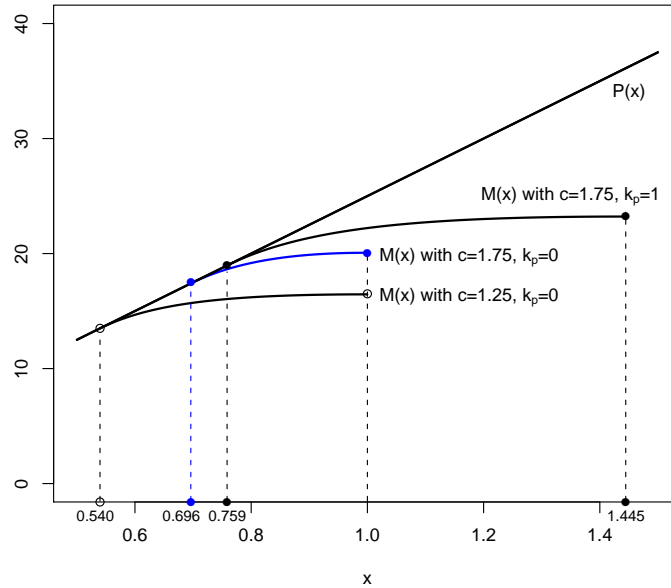
c	k_p	x^0	x_*	x_{**}	x^*	DOV	POV	OV
1.25	0.000	0.714	0.554	0.540	1.000	1.662	0.381	2.043
	0.250	0.714	0.554	0.552	1.475	1.662	0.052	1.714
	0.500	0.714	0.554	0.554	3.514	1.662	0.001	1.663
	0.518	0.714	0.554	0.554	∞	1.662	0.000	1.662
1.75	0.000	1.000	0.776	0.696	1.000	2.327	2.603	4.930
	0.250	1.000	0.776	0.730	1.157	2.327	1.344	3.671
	0.500	1.000	0.776	0.742	1.250	2.327	0.930	3.257
	0.750	1.000	0.776	0.752	1.343	2.327	0.656	2.983
	1.000	1.000	0.776	0.759	1.445	2.327	0.456	2.782
	1.250	1.000	0.776	0.764	1.565	2.327	0.304	2.630
	1.500	1.000	0.776	0.769	1.719	2.327	0.188	2.514
	1.750	1.000	0.776	0.772	1.939	2.327	0.100	2.427
	2.000	1.000	0.776	0.774	2.322	2.327	0.039	2.366
	2.250	1.000	0.776	0.776	3.599	2.327	0.004	2.331
	2.327	1.000	0.776	0.776	∞	2.327	0.000	2.327

First, we consider the effect of the prepayment penalty k_p on the borrower's prepayment decision. When there is a positive penalty, the borrowers do not prepay right away at $x^* = 1$, i.e., the prepayment point is higher than 1, i.e., $x^* > 1$. Also the table shows that the bigger the penalty k_p is, the higher x^* is. For example, when $c = 1.75$, without prepayment penalty ($k_p = 0$), the borrower prepays her loan at $x = 1$. But with $k_p = 0.5$, the borrower prepays at the higher x , i.e., at $x = 1.25$ and at $x = 2.322$ with $k_p = 2$. The borrowers hold mortgages longer and prepay at the higher x since the high prepayment penalty implies the high price of

the prepayment. Therefore, the prepayment penalties are effective deterrents of prepayment. Imposing the prepayment penalty on the subprime mortgage makes refinancing difficult. The table also shows that the value of prepayment option is reduced by the increase of the prepayment penalty. Particularly for the subprime mortgage, the reduction of prepayment option value by the prepayment penalty is drastic. For example, if the lender impose the penalty $k_p = 1$ when $c = 1.75$, the prepayment option value decreases by 2.147, which is more than two years worth of the housing services. If the lender imposes the maximum prepayment penalty k_p^* , the prepayment option value disappears completely and the total option value is just the default option value. The prepayment is not an available option anymore for the borrower since it is prohibitively costly. The borrower is trapped in the mortgage contract with a high periodic payment. Thus, the mortgage contract with an extremely high prepayment penalty is the same as the mortgage contract without the prepayment option. Thus, the prepayment does not happen, i.e., $x^* = \infty$. Refinancing is not possible for the subprime borrowers.

The prepayment penalty affects not only the prepayment decision but also the default decision. Table (6) shows the effect of the prepayment penalty, k_p on the default point, x_{**} : The higher k_p is, the higher x_{**} is. This is because the value of the mortgage depends on not only the value of default option but also on the value of the prepayment option. When the prepayment penalty is big, the value of prepayment is small, and so is the mortgage value. The prepayment penalty takes the value of prepayment option away from the mortgage borrowers. This reduction of the option value induced by the penalty affects the default decision of the subprime borrowers. Thus, borrowers default more quickly or at the higher default point when the prepayment penalty is large. That is, the expected optimal default time, $E(T(x_{**}))$ becomes smaller. For instance, when c is 1.75, where k_p is 0.5, the option value is 3.257 and when k_p is 2, the option value is 2.366. Accordingly, for $k_p = 0.5$ and 2, the default points are

Figure 5: Mortgage value with and without the prepayment penalty



0.742 and 0.774 respectively. Figure (5) shows that the loans with prepayment penalties are significantly more likely to default than those without prepayment penalties. When $c = 1.25$ without the penalty, which we consider being as the prime mortgage, the default point is as low as 0.540 and for the subprime mortgage, $c = 1.75$ with the penalty $k_p = 1$, x_{**} is 0.759. The difference is as large as 0.219.¹⁵ As k_p increases, x_{**} converges to the x_* . Thus, at k_p^* , the default point in the model with prepayment option is same as the default point in the model without the prepayment option, i.e., $x_{**} = x_*$.

For the financially distressed subprime borrower, defaulting is an optimal choice when the prepayment penalty, the cost of prepayment, is high relative to the cost of default. In this sense, the default option is a substitute to the prepayment option. When house prices rise, subprime borrowers can accumulate home equity from the price appreciation. Thus,

¹⁵Roberto G. Quercia, Michael A. Stegman, and Walter R. Davis (2007) showed that, controlling for other risk factors, the odds of foreclosure for loans with prepayment penalties were about 20 percent higher than for loans without prepayment penalties.

a financially distressed borrower could avoid default by prepaying the loan even if there is a prepayment penalty. Hence there was a sustained high prepayment rates of the subprime mortgages in the first part of this decade.¹⁶ However, the prepayment option is no longer available when prices depreciate since the optimal prepayment point x^* with a positive prepayment penalty is always above 1. Prepaying at 1 or below 1 is not an optimal prepayment decision as shown in table (6) and figure (5). Thus, the subprime borrowers can be trapped in the mortgage contract with high periodic payment. Also the default point of the subprime borrower is higher with a positive prepayment penalty. This high default point makes the subprime borrower much more vulnerable to the price depreciation. Therefore, this inability to refinance the mortgage loans and the high default point induced by the prepayment penalty translated into the high rate of mortgage defaults and foreclosures. Consequently, borrower default began to increase as house price started to depreciate in 2006.

7 Conclusion

We characterize the mortgage as a fixed income security with two options, the default option and the prepayment option. By transforming this characterization of mortgages into the optimal stopping time framework, we are able to examine the effects of prepayment penalties on the termination decision of subprime borrowers.

We identify two effects of the prepayment penalty on subprime borrowers' mortgage termination decisions. One is that a high prepayment penalty deters the prepayment. As shown in the model, the prepayment penalty makes the optimal prepayment point higher. In other words, it makes refinancing more difficult. The prepayment penalty affects not only the pre-

¹⁶The prepayment rate of the fixed rate mortgages up to five years after the origination dates are 50%, 55%, 60%, 68%, 70% in years 1998, 1999, 2000, 2001, 2002, 2003 respectively. The prepayment rate of the adjustable mortgage rates are much higher than these numbers.

payment decision but also the default decision of subprime borrowers. The options embedded in the mortgage contracts add economic values to the mortgages since these options provide a partial protection from the volatility of the house prices. In particular, the value of the prepayment option for the subprime mortgage borrower is significant. The prepayment penalty increases the likelihood of default by subprime borrowers because the penalty reduces the option values of the mortgages.

By these two effects of prepayment penalty on mortgage termination decision, the default decision of the borrower becomes more susceptible to the house price depreciation. When house price depreciates, the subprime borrowers are trapped in the mortgage contract with high periodic payments. Thus, financially distressed borrowers have only one option: default. With an enough house price fall, even if subprime borrowers are not financially distressed, defaulting becomes an optimal decision.

The existence of the prepayment penalty in subprime mortgage contract amplifies the effect of housing price decline on mortgage default. This is the one of the reason why there was a sharp increase of the default and foreclose rates in the subprime mortgage market after 2006.

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